Regular Mesh Construction Algorithms using Regular Handles

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Abstract

This paper presents our recent theoretical results on high genus modeling. We introduce a new concept called regular handles. Using regular handles it is possible to increase genus without increasing the number of vertices.

Using regular handles a wide variety of mesh structures can be constructed. One of the usages of regular handles is to construct families of regular meshes, which is useful to create a wide variety of high genus mesh structures.

1. Introduction

In this paper, we introduce the concept of regular handles. Regular handles create handles without increasing the number of vertices. Their only effect is to increase the vertex valence and the genus of the mesh.

Remeshing schemes based on regular handles are not subdivision type since the number of vertices does not change and genus can increase. The construction algorithms using regular handles provide a new modeling paradigm to create a wider variety of meshes.

1.1. Topologically Regular Meshes

Akleman and Chen have recently introduced the concept of topological regularity as all faces and vertices have the same combinatorial property; i.e having the same size and same valence, respectively [2]. The valence of a vertex is defined as the number of edge-ends that emanates from that vertex. The face size is counted as the number of edge sides (also called half-edges [19]) belonging to a face. In other words, two sides of a same edge can belong to the same face and that edge will be counted twice. A regular mesh is represented with a triple (n,m,g) where n is the face size, m is the vertex valence and g is the genus of the surface. For $g = 0$, regular meshes include regular platonic solids, all two sided polygons. For $g = 1$ regular meshes include regular tilings of infinite plane.

Akleman and Chen also showed that there exist infinitely many regular meshes. They have also proved that for given n, m, and $g > 1$, a regular mesh exists for (n,m,g), then we can construct regular meshes $(n,m,k(g-1)+1)$ for any $k \geq 1$ from (n,m,g). Based on their proof, it is important to construct (n,m,g) for any given n,m and smallest possible g. Among the regular meshes, $(n,m,2)$'s are particularly important since if a regular mesh $(n,m,2)$ exists, it is possible to construct (n,m,g) for any given g. Akleman and Chen have provided construction algorithms for two families of regular meshes $(4,g+3,g)$ and $(3,g+5,g)$ that include genus-2 primary meshes ($n \leq m$) $(3,7,2)$ and $(4,5,2)$. In addition, they have shown the existence of genus-2 primary meshes, $(4,6,2)$ and $(5,5,2)$ by drawing figures [2].

In this paper, we develop a general procedure based on regular handles. Our procedure allows us to greatly extend *"regular mesh families"*. We provide 14 regular mesh families that includes all genus-2 primary regular meshes: $(3,7,2)$, $(3,8,2)$, $(3,9,2)$, $(3,10,2)$, $(3,12,2)$, $(3,18,2)$, $(4,5,2)$, $(4,6,2)$,$(4,8,2)$, $(4,12,2)$ and $(5,5,2)$, $(5,10,2)$, $(6,6,2)$ and $(8,8,2)$. Our regular mesh families are constructed by adding the regular handles to an initial regular mesh M_0. By using the same procedure iteratively we construct a series of regular meshes $M_0, M_1, M_2, \dots M_n$ as

$$\begin{array}{lllll} M_1 & = & M_0 + kH & & \\ M_2 & = & M_0 + 2kH & = & M_1 + kH \\ \cdots & & \cdots & & \cdots \\ \cdots & & \cdots & & \cdots \\ M_n & = & M_0 + nkH & = & M_{n-1} + kH \end{array}$$

where $+kH$ is the operation that adds k regular handles. The number of vertices does not increase by adding regular handles. All regular meshes, $M_0, M_1, M_2, \dots M_n$, share the same initial vertices. On the other hand, the number of faces, edges and genus can increase in each iteration.

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1.2. Motivation for Studying Regular Meshes

The motivation to study regular meshes comes from understanding the power of various mesh modeling approaches. A recent work [23] suggested that subdivision schemes [26] are more powerful than various fractal schemes such as Iterated Function Systems (IFS) [6, 18]. Another recent result [5] further showed the limitations of subdivision schemes. Akleman and Chen's preliminary work on regular meshes implicitly showed that vast majority of regular meshes cannot be created by subdivision schemes (There are, of course, some exceptions such as $(4,4,1)$, $(6,3,1)$ and $(3,6,1)$) [2]. Moreover, it is possible to create dodecahedron $(3,5,0)$ from tetrahedron $(3,3,0)$ by using pentagonal conversion algorithm [5]. Simplest subdivision [22] can create octahedron $(3,5,0)$ from tetrahedron $(3,3,0)$.

Another motivation for studying regular meshes is texture mapping. The regular meshes in the form of $(3,m,g)$ and $(4,m,g)$ provide nice triangular and quadrilateral subdivisions of high genus surfaces. These quadrilateral or triangular patches can seamlessly be covered by aperiodic tiles [24, 20, 10, 4]. $(3,m,g)$ and $(4,m,g)$ can also provide a framework to describe control meshes for patch modeling [25]. We also think that regular meshes will eventually be useable for topological simplification of meshes. Regular meshes can also be useful for morphing high genus surfaces from one to another. We note that there has been extensive literature in mathematical research on the related topics [21, 12, 16]. For example, regular meshes on surfaces of genus 1 and 2 have been investigated by Brahana [8]. Three regular polyhedra for infinite genus, $(6,4,\infty)$, $(4,6,\infty)$ and $(6,6,\infty)$ are discovered by Coxeter [11, 14, 9, 15]

2. Previous Work

Polygonal meshes, which are the most commonly used representations in computer graphics applications, represent complicated surfaces by subdividing them into simpler surfaces, which are called *faces* [19, 7, 1]. This subdivision idea even exists in real life. We often create complicated surfaces by combining simpler patches. Formally, a *mesh* on a 2-manifold S specifies an embedding $\rho(G)$ of a graph G on the 2-manifold S. Each connected component of $S-\rho(G)$ plus the bounding edges in G makes a *face* of the mesh. Following the conventions adopted by many other researchers [17, 19], we allow a mesh to have multiple edges (i.e., more than one edge connecting the same pair of vertices) and self-loops (i.e., edges with both ends at the same vertex). The mesh is *cellular* if the interior of each face is homeomorphic to an open disk. Non-cellular meshes have also been studied in the literature but the corresponding modeling algorithms are more complicated [17, 19].

In topological mesh modeling [1, 3], we only consider cellular meshes and ignore any geometric condition. We assume that faces and edges can have any shape. With this assumption, although we lose the advantages coming from planarity/linearity of faces and edges, we gain a lot. Without geometric constraints, it is not only possible to develop simpler and faster algorithms for interactive modeling; it is also possible to represent a wide variety of shapes.

Along this line of thinking, the regular meshes are potentially very useful modeling tool since they provide essential subdivisions of surfaces for any given genus. One particular example is $(4g,4g,g)$ that is widely used in topology to cut a genus-g surface to a $4g$-gon polygon. Ferguson et al. used $(4g,4g,g)$ to create high genus smooth surfaces [13].

3. Regular Handles

We start with an example to motivate our construction. Consider Figure 2. Suppose that we are given a dodecahedron D, which is a regular mesh of type $(5,3,0)$. Pick any pair of faces on D (which are pentagons), and connect them by a deformed prism. Note that all "new faces" in the deformed prism are quadrilaterals. Moreover, the two chosen faces on D have disappeared after this construction. Therefore, if we pair all the faces on D (note that the dodecahedron D has 12 faces), and connect each face pair using such a "quadrilateral handle", we get a new mesh D' in which all faces are of size 4. Moreover, since the dodecahedron D is a regular mesh in which each vertex has valence 3 and belongs to exact 3 face corners, all vertices in the new mesh D' after the construction have valence 6. Finally, since connecting two faces in D using a deformed prism increases the mesh genus by 1 and there are six pairs of faces in D, the new mesh D' has genus 6. In conclusion, the new mesh D' is a regular mesh of type $(4,6,6)$ (See Figure 3).

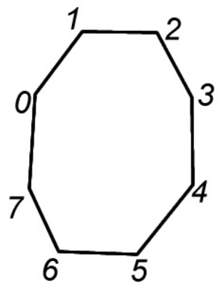
3.1. Definition

The above process suggests a new procedure for constructing new regular meshes from existing regular meshes. For this, we introduce the following definition.

Definition Let M be a mesh, and let $S=\{c_0,\ldots,c_{r-1}\}$ be a set of face corners in M. A *regular handle H on the face corner set S* is a structure added to M by inserting a sequence of new edges[1] $\{e_0,\ldots,e_{p-1}\}$ to M such that

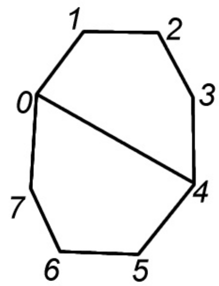
(1) the ends of the new edges are all inserted to the face corners in S such that exactly $2p/r$ edge ends are inserted into each face corner in S (in particular, $2p$ must be divisible by r);

1 See [1, 3], for formal definition of insert edge operation

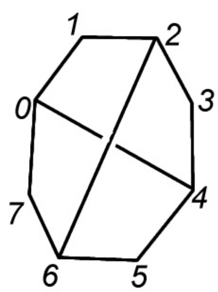


Initial octagonal face

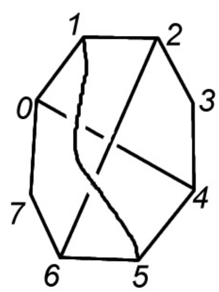
$f_0 = (0, 1, 2, 3, 4, 5, 6, 7)$



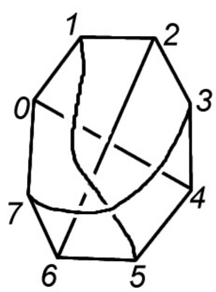
Inserting an edge splits octagon to two hexagons
$f_1 = (0, 1, 2, 3, 4)$
$f_2 = (0, 4, 5, 6, 7)$



Inserting another edge combines hexagons and creates a 14-gon handle
$(0, 1, 2, 6, 7, 0, 4, 5, 6, 2, 3, 4)$



Inserting an edge splits 14-gon to two octagons
$f_4 = (0, 1, 5, 6, 2, 3, 4)$
$f_5 = (0, 4, 5, 1, 2, 6, 7)$



Inserting another edge combines octagons and creates another handle which is a 16-gon
$f_6 = (0, 1, 5, 6, 2, 3, 7, 0, 4, 5, 1, 2, 6, 7, 3, 4)$

Figure 1. The effect of inserting a sequence of new edges by combining and spitting faces.

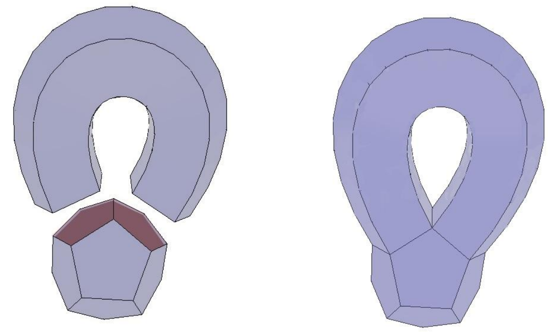


Figure 2. A quadrilateral handle (a deformed pentagonal prism) that connects two faces of a dodecahedron, increasing the genus by one.

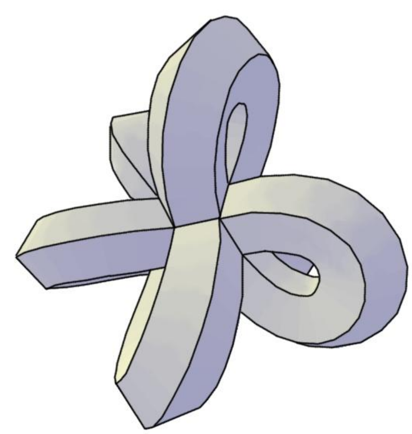


Figure 3. A regular mesh $(4, 6, 6)$ constructed by adding 6 quadrilateral handles to a dodecahedron.

(2) all new faces resulting from this process (i.e., the faces that are not faces in the original mesh M) have the same size.

A regular handle H on a face corner set S in a mesh M is called an (n, m, g)-*handle* if all resulting new faces have size n, exactly m edge ends are inserted into each face corner in the set S, and the genus of the new mesh is g larger than the original mesh. For example, the regular handle shown in Figure 2 is a $(4, 1, 1)$-handle.

Remark 1. The simplest regular handle is a $(k+1, 1, 0)$-

handle on two corners of a face of size $2k$ that inserts an edge between the two corners and splits the face into two faces of size $k+1$ (see the second figure in Figure 1, in this case, the mesh genus is not changed). A slightly more complicated regular handle is a $(k+h+2,1,1)$-handle on two corners of two different faces of size k and h, respectively, that inserts an edge in the two face corners and merges the two faces into a face of size $k+h+2$ (see the third figure in Figure 1, in this case the mesh genus is increased by 1). Moreover, by the definition, two consecutive edge insertions on four face corners in a face F of size k, in which the first splits the face F into two faces (arbitrarily) and the second merges the two new faces into a single face, also constitute a regular handle of type $(k+4,1,1)$ (see Figure 1 for illustrations).

Remark 2. Although not the simplest, the most widely used regular handles are $(4,1,1)$-handles that connect two faces of the same size on a mesh M, as shown in Figure 2. Such a $(4,1,1)$-handle is obtained by first inserting an edge that merges two corners in the two faces then inserting $p-1$ edges that connect the remaining vertices in the faces in the same order (where p is the size of the faces). The flattened version of such a $(4,1,1)$-handle that connects two p-sided faces is shown in Figure 7.

Remark 3. The following fact is useful for our further discussion. Suppose that a regular handle H is constructed on a face corner set S. If F is a face that has at least one face corner in S, then after the construction of the regular handle H, the face F disappears in the new mesh.

Remark 4. Regular handles can connect more than two faces as shown in Figures 4 and 5.

3.2. Regular Mesh Constructions with Regular Handles

The concept of regular handles and the above remarks suggest a very general approach for constructing a variety of regular meshes from existing regular meshes.

For example, suppose that we are given a regular mesh M of type (n,m_1,g_1) and we want to construct other types of regular meshes of the same face size n. We may consider constructing a sequence of regular handles of type (n,m_2,g_2): H_1 on face corner set S_1, H_2 on face corner set S_2, …, H_r on face corner set S_r such that each vertex v in M contributes exactly the same number h of face corners in the face corner sets S_1, …, S_r. Suppose that such a regular handle sequence on M exists, and let the new mesh after adding these regular handles to M be M'. Then the face size of the mesh M' is still n. Moreover, since exactly m_2 edge ends are inserted in each face corner in the sets S_1, …, S_r and the number h of face corners on each vertex is the same for all vertices in M, the valence of all vertex in M is increased by the same number hm_2. Therefore, the new

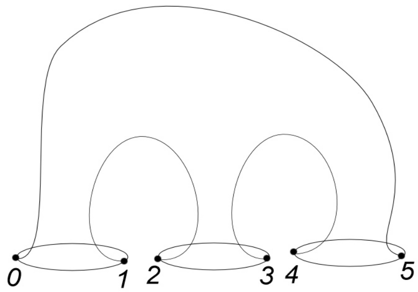


Figure 4. A $(6,1,2)$ handle that can connect 3 two-sided faces. In this example, initially we have three faces $f_0=(0,1)$, $f_1=(2,3)$ and $f_2=(4,5)$. After adding regular handle, f_0, f_1 and f_2 disappear and two new hexagonal faces are created: $f_3=(0,1,2,3,4,5)$ and $f_3=(5,4,3,2,1,0)$

mesh M' is a regular mesh. Since each regular handle increases the mesh genus by g_2, the genus of the new mesh M' is equal to g_1+g_2r. In conclusion, from the (n,m_1,g_1)-mesh M, we have constructed a regular mesh M' of a new type (n,m_2h,g_1+g_2r), which has the same face size.

As another example, suppose that we are given a regular mesh M of type (n_1,m_1,g_1) and we want to construct regular meshes of a different face size n_2. Then we may consider constructing a sequence of regular handles of type (n_2,m_2,g_2): H_1 on face corner set S_1, H_2 on face corner set S_2, …, H_r on face corner set S_r, such that each vertex v in M contributes exactly the same number h of face corners in the sets S_1, …, S_r, *and* each face in M has at least one corner in the sets S_1, …, S_r. If such a regular handle sequence exists, then, similar to the above analysis, we will obtain a regular mesh of type (n_2,m_2h,g_1+g_2r) (note that by Remark 3 and since every face in M has at least one corner in the sets S_1, …, S_r, all faces in the original mesh M disappear so that all faces in the new mesh M' have size n_2).

A special case is that for the regular mesh M of type (n_1,m_1,g_1), each face corner appears exactly once in the corner sets S_1, …, S_r. In this case, the regular handle construction results in a regular mesh of type $(n_2,(m_2+1)m_1,g_1+g_2r)$. This can be seen from Figure 2 as an example. The original mesh D is a regular mesh of type $(5,3,0)$, with $n_1=5$, $m_1=3$, and $g_1=0$. By adding 6 regular handles of type $(4,1,1)$ (thus, $n_2=4$, $m_2=1$, and $g_2=1$), which connect different faces in D, we obtain a regular mesh D' of type $(4,(1+1)\cdot 3,0+1\cdot 6)=(4,6,6)$.

The general procedure for our construction algorithms

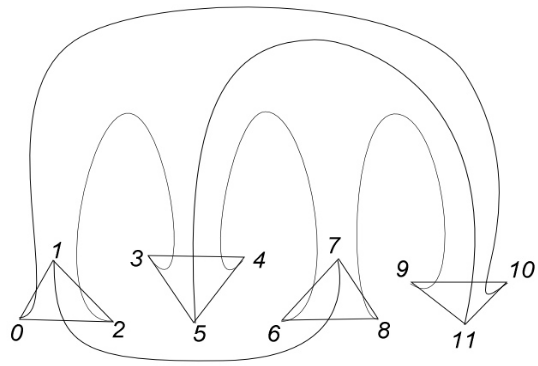


Figure 5. A $(6,1,3)$ handle connects 4 triangular faces. In this example, we have initially four triangular faces $f_0 = (0,1,2)$, $f_1 = (3,4,5)$, $f_2 = (6,7,8)$ and $f_3 = (9,10,11)$. After adding the regular handle, triangles disappear and four new hexagonal faces are created: $f_4 = (0,10,11,5,3,2)$, $f_5 = (0,1,7,8,9,10)$, $f_6 = (1,7,6,4,3,2)$ and $f_7 = (5,11,9,8,6,4)$

operates in exactly the same manner as above. We start with a regular mesh and add regular handles until we get the desired regular mesh.

Remark 5. Although we do not cover in this paper, regular meshes can also be obtained from semi-regular meshes. Figure 6 shows an example.

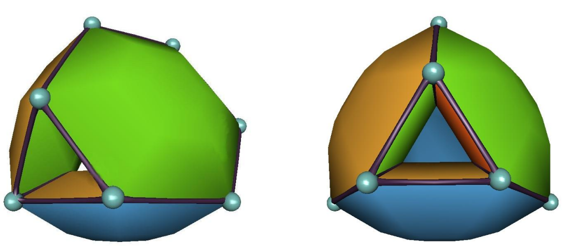


Figure 6. A $(6,4,3)$ mesh that is created by connecting four triangles of truncated tetrahedron with a $(6,1,3)$ handle.

3.3. Regular Handle Constructions

In this section, we show how to construct a simple set of regular handles that connects two faces. Suppose that we start with a $(4,1,1)$-handle H_1 connecting two faces of the same size in a mesh M, as given in Figure 7. We can split the quadrilaterals in the $(4,1,1)$-handle H_1 to obtain a $(3,2,1)$-handle H_2, if we insert an edge between two corners of the same quadrilateral in the $(4,1,1)$-handle H_1, as shown in Figure 8.

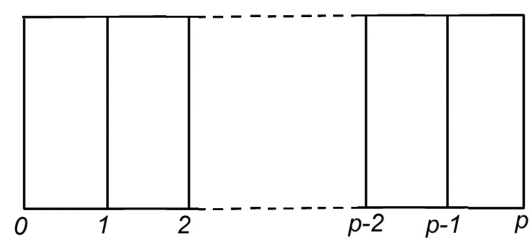


Figure 7. A flattened $(4,1,1)$-handle. Edges 0 and p are the same.

Now starting from the $(3,1,1)$-handle H_2 obtained from the above process, if we insert an edge between two consecutive triangular faces in H_2, as shown in Figure 9, these two triangular faces are merged into an octagon and the mesh genus is increased by 1. If we apply this merging operation to each pair of consecutive triangular faces in H_2, as shown in Figure 9, we obtain a $(8,3,p+1)$-handle with octagonal faces (recall that p is the size of the two faces in the original mesh M).

We can continue in the same way starting with the $(8,3,p+1)$-handle by splitting we can obtain a $(5,4,p+1)$-handle. This whole operation can be generalized to two types of handles that are created from each other as follows:

- $(4k+4,2k+1,kp+1)$-handles that are created by applying combine operation to $(2k+1,2k,(k-1)p+1)$-handles.
- $(2k+3,2k+2,kp+1)$-handles that are created by applying split operation to $(4k+4,2k+1,kp+1)$-handles.

This procedure can create all handles with odd face sizes and even face sizes that is divisible by 4. However, it does not create even face sizes that is not divisible by 4. The even face sizes not divisible by 4 can be created starting from $(4,1,1)$-handle with a slightly different procedure. As shown in Figure 11, a $(6,2,p/2+1)$-handle can be created by first combining then splitting quadrilaterals of a $(4,1,1)$-handle with two insert edge operations (here we assume that p is an even number). Recursive applications of this operation create $(2k+4,k+1,kp/2+1)$-handles from $(2(k-1)+4,(k-1)+1,(k-1)p/2+1)$-handles for $k \geq 1$.

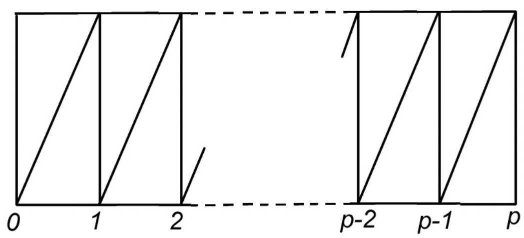


Figure 8. The $(3,2,1)$-handle created by split operation that splits quadrilaterals of $(4,1,1)$-handle with an insert edge operation.

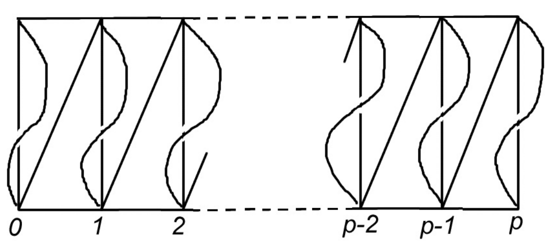


Figure 9. The $(8,3,p+1)$-handle created by a combine operation that combines the neighboring triangles of $(3,2,1)$-handle with an insert edge operation.

Remark 6. As discussed earlier, regular handles are not limited handles that combine two faces. It is possible to combine any number of faces with regular handles. However, those handles are beyond the scope of this paper. For simple examples, see Figures 4 and 5. Figure 6 is a regular mesh created by a regular handle that combines more than 2 faces.

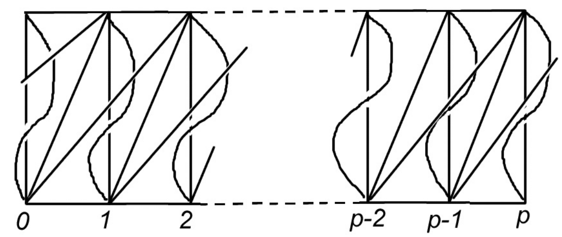


Figure 10. The $(5,4,p+1)$-handle created by splitting octagons of $(8,3,p+1)$-handle with an insert edge operation.

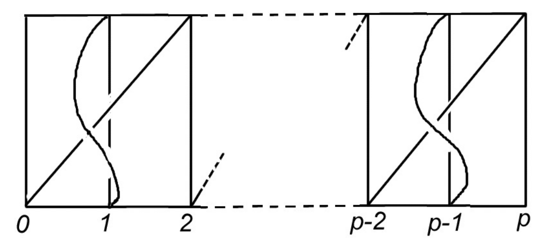


Figure 11. The $(6,2,p/2+1)$-handle created by first combining and then splitting quadrilaterals of $(4,1,1)$-handle with two insert edge operations.

4. Primary Regular Mesh Families with Constant Face Size

Using regular handles that connect two faces, it is easy to develop construction algorithms for primary regular mesh families with constant face sizes. To develop construction algorithms for these families, we need only one additional operation, called **Edge_Split**. This operation splits an edge and creates a two-gon. The operation is useful to create two-gons and combining them with regular handles.

We first identify primary regular mesh families when the number v of vertices and face size n are fixed constants. Since the number v of vertices does not increase with adding regular handles, identification of these families will also provide us the corresponding construction algorithms. Based on Euler's equation (see Appendix) for the fixed values for face size n and the number v of vertices, the relationship between vertex valence m and genus g will be computed by the following equation:

$$m = \frac{2n(2g-2+v)}{v(n-2)} \quad (1)$$

The integer solutions for this equation (regardless of the g values) will provide us potential regular mesh families. The reason we call them potential is that having an integer solution does not guarantee the existence of the corresponding regular meshes, unless we have developed a construction algorithm.

From this equation, It can be seen that only for $n = 3,4,6$ the equation can hold for all values of g. However, all integer solutions must also provide integer number of faces. One of the solutions of this equation for $n = 6$ and $v = 1$ gives the family $(6,6g-3,g)$. But, the regular meshes in this family only exists for odd values of g since it does not have integer number of faces for even values of g. Although, it is interesting to study such families, in this paper, we focus on the families that exists for all g values larger than 0.

Once the family is identified, it is easy to develop a construction algorithm. For instance, let $n = 4$ and $v = 8$. For this particular case, the regular mesh family is $(4,g+3,g)$. We identify the regular mesh with smallest genus by choosing smallest g possible. In this case $g = 0$ and the simplest regular mesh is a cube $(4,3,0)$.

Since $(4,g+3,g)$ consists of quadrilateral meshes, we have to use regular handle $(4,1,1)$ to create the members of this family. This regular handle increase the genus by one and increase the valence by one. This is exactly the requirement coming from the family. When genus increases by one valence also increases by one. So, the process in this case is straightforward. If we apply a rectangular handle once, we get $(4,4,1)$. If we apply it twice we get $(4,5,2)$ and so on.

Once we have identified what type of handle and how many times is needed to be used, the rest is straightforward. We identify the faces to be connected by handle that increase the valences of all vertices based on requirement.

Using this approach, we are able to develop algorithms to construct all regular mesh families that can be created by adding the same regular handle. These families include all quadrilateral and triangular regular meshes for $g = 2$; which are (4,5,2), (4,6,2), (4,8,2), (4,12,2), (3,7,2), (3,8,2), (3,9,2), (3,10,2) and (3,12,2). Each one of these belongs to one the following regular mesh families.

1. (4, g+3, g): This group always have 8 vertices. These regular meshes are created by following iterated sequence:

   Start with a cube (4,3,0) and add one $(4,1,1)$-handle between any two non-touching faces of the cube. Continue with the same way. This creates the following series of regular meshes (4,3,0), (4,4,1), (4,5,2) and so on... (See Figure 12)

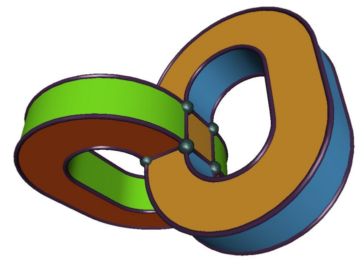


Figure 12. Regular mesh $(4,5,2)$ that is obtained by adding two $(4,1,1)$-handles to a cube.

2. (4, 2g+2, g): This group always have 4 vertices. These regular meshes are created by following iterated sequence:

   Start with a quadrilateral-manifold (4,2,0) (i.e., a two-sided quadrilateral) and add one $(4,1,1)$-handle between the two faces of the quadrilateral-manifold. Continue with the same way. This creates the following series of regular meshes (4,2,0), (4,4,1), (4,6,2) and so on...

3. (4, 4g, g): This group always have 2 vertices (valid only $g \geq 1$). These regular meshes are created by following iterated sequence:

   Start with a donut (4,4,1) with two vertices and add one $(4,1,1)$-handle between the two faces of donut. Continue with the same way. This creates the following series of regular meshes (4,4,1), (4,8,2). (4,12,3) and so on...

4. (4, 8g-4, g): This group always have 1 vertex (Valid only $g \geq 1$). These regular meshes are created by following iterated sequence:

   Start with a donut (4,4,1) with one vertex and split two edges. Add one $(4,1,1)$-handle between the newly created two two-gons. Continue with the same way. This creates the following series of regular meshes (4,4,1), (4,12,2). (4,20,3) and so on...

5. (3, g+5, g): This group always have 12 vertices. These regular meshes are created by two iterated sequences:

   (a) Start with icosahedron $(3,5,0)$ and add two $(3,2,1)$-handle in each iteration between any four non-touching faces of icosahedron. This creates the following series of regular meshes (3,5,0), (3,7,2), (3,9,4) and so on...

   (b) Start with a donut $(3,6,1)$ with 12 vertices and add two $(3,2,1)$-handle in each iteration between any four non-touching faces of

(3,6,1). This creates the following series of regular meshes (3,6,1), (3,8,3), (3,10,5) and so on...

6. (3, 2g+4, g): This group always have 6 vertices. These regular meshes are created by following iterated sequence:
   Start with a octahedron $(3,4,0)$ and add a $(3,2,1)$-handle in each iteration between two non-touching triangles. This creates the following series of regular meshes (3,4,0), (3,6,1), (3,8,2) and so on...
7. (3, 3g+3, g): This group always have 4 vertices. These regular meshes are created by following iterated sequence:
   Start with a tetrahedron $(3,3,0)$. First convert two non-touching edges of tetrahedron to two-gons. Then add one $(3,2,1)$-handle between these two two-gons. Continue with the same way. This creates the following series of regular meshes (3,3,0), (3,6,1), (3,9,2) and so on...
8. (3, 4g+2, g): This group always have 3 vertices. These regular meshes are created by following iterated sequence:
   Start with a triangle manifold (3,2,0) and add one $(3,2,1)$-handle between the two opposing faces of the triangle manifold. Continue with the same way. This creates the following series of regular meshes (3,2,0), (3,6,1), (3,10,2) and so on...
9. (3, 6g, g): This group always have 2 vertices. (Valid for $g \geq 1$) These regular meshes are created by following iterated sequence:
   Start with a donut $(3,6,1)$ with 2 vertices. Add one $(3,2,1)$-handle between any two triangles (These two triangles together must have each vertex thrice). Continue with the same way. This creates the following series of regular meshes (3,6,1), (3,12,2) and (3, 18, 3) and so on...
10. (3, 12g-6, g): This group always have one vertex. (Valid for $g \geq 1$) These regular meshes are created by following iterated sequence:
   Start with a donut $(3,6,1)$ with 1 vertex. Add one t$(3,2,1)$-handle between any two triangles. Continue with the same way. This creates the following series of regular meshes (3,6,1), (3,18,2) and (3, 30, 3) and so on...

5. Primary Regular Mesh Families with Constant Number of Faces

To create these families, we assume that the number of faces stays the same in each iteration along with the number of vertices. Choosing the number of faces, f, constant means that both n and m increases in sync. The families will be identified by choosing a value for v and assuming $kv = f$ where k is any integer. $kv = f$ also means $kn = m$. Given these ratios between v, f and n, m, the relationship between face size and genus will be computed by the following equation:

$$n = \frac{4(g-1)+2(k+1)v}{kv} \tag{2}$$

There are only four families in this category and they can be created by using simplest regular handles that consists of only two edge insertions; one for face split and the other for face connect.

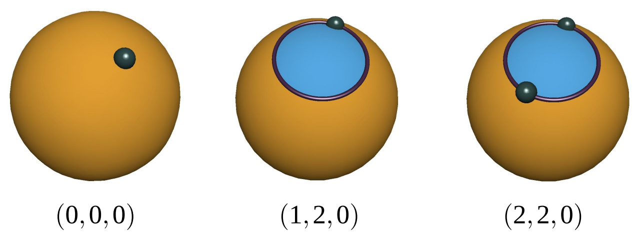


Figure 13. The regular meshes that are used as initial meshes for regular mesh families with constant number of faces and vertices.

1. (2g+1, 4g+2, g): This group always have 1 vertex and 2 faces (k=2). This creates the following series of regular meshes (1,2,0), (3,6,1), (5,10,2) and (7, 14, 3) and so on... The number of edges is $2g+1$ and the number of edge insert required in each iteration is 2. Since the genus increases by one, the first edge insert will combine 2 faces, the other one will again split.
2. (4g, 4g, g): This group also always have 1 vertex and 1 face (k=1). This creates the following series of regular meshes (4,4,1), (8,8,2) and (12, 12, 3) and so on... The number of edges is $2g$ and the number of edge insert required in each iteration is 2. In this case, the first edge insert will split the face into two, the next one will combine them again.
3. (2g+2, 2g+2, g): This group always have 2 vertices and 2 faces (k=1). This creates the following series of regular meshes (2,2,0), (4,4,1), (6,6,2) and (8, 8, 3) and so on... The number of edges is $2g+2$ and the number of edge insert required in each iteration is 2. In this case, the first edge insert will combine 2 faces, the other one will again split. Since we have 2 vertices, valence increase is just 2 in each iteration

4. (g+3, g+3, g): This group always have 4 vertices and 4 faces (k=1). This creates the following series of regular meshes (3,3,0), (4,4,1), (5,5,2) and (6, 6, 3) and so on... The number of edges is $2g+6$ and the number of edge insert required in each iteration is 2. In this case, the first edge insert will combine any 2 faces of initial 4, the other one will again split them. Since we have 4 vertices each edge insert will use different vertices and valence increase is just 1 in each iteration. (See Figure 14 for regular mesh $(5,5,2)$)

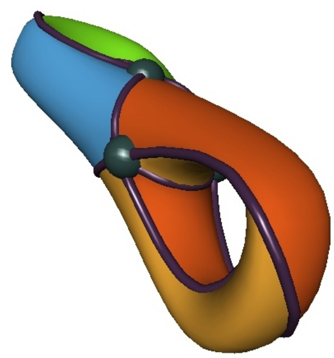


Figure 14. Regular mesh $(5,5,2)$ that is created starting from a tetrahedral mesh.

6. Conclusion

In this paper, we have introduced the concept of regular handles and identified regular mesh families. Using regular handles we have developed construction algorithms for regular mesh families that provide a new modeling paradigm to create a wider variety of meshes.

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A. Appendix: The Relationship Between Genus, Face Size, Valence, and the number of Edges, Vertices and Faces in Manifold Meshes

In topological mesh modeling, our only concern is mesh structure; how faces, edges and vertices are related with each other. Euler equation is the fundamental equation that gives the relationship between the number of faces, f, the number of edges, e, and the number of vertices v. Using Euler-Poincare equation, without using geometric properties, we can identify some essential properties of manifold meshes. Euler-Poincare equation is given as follows:

$$f - e + v = 2 - 2g \tag{3}$$

where g is the total number of handles in the surface, called genus.

Using Euler-Poincare equation, it is possible to systematically search for regular meshes. First note that if all faces have the same number of sides n and all vertices has the same valence m, we can obtain the following relationships:

$$nf = 2e \tag{4}$$

$$mv = 2e \tag{5}$$

If we plug in these relationships in Euler-Poincare equation, we obtain a simplified equation

$$\left(\frac{1}{n} + \frac{1}{m} - \frac{1}{2}\right) e = 1 - g. \tag{6}$$

The integer solutions of these equations for any given n, m, g give us an idea about regular manifold meshes for that triplet. But, note that having integer solutions to equation (6) alone does not prove existence of the regular mesh. If we rearrange the Euler-Poincare equation, we find the following equations for e, v, and f.

$$e = \frac{2nm}{nm - 2n - 2m}(g - 1) \tag{7}$$

$$f = \frac{4m}{nm - 2n - 2m}(g - 1) \tag{8}$$

$$v = \frac{4n}{nm - 2n - 2m}(g - 1) \tag{9}$$

This formula is useful for $nm - 2n - 2m \neq 0$ which in fact corresponds $g = 1$ case.